Week 05: Impulse Response and Laplace Transform

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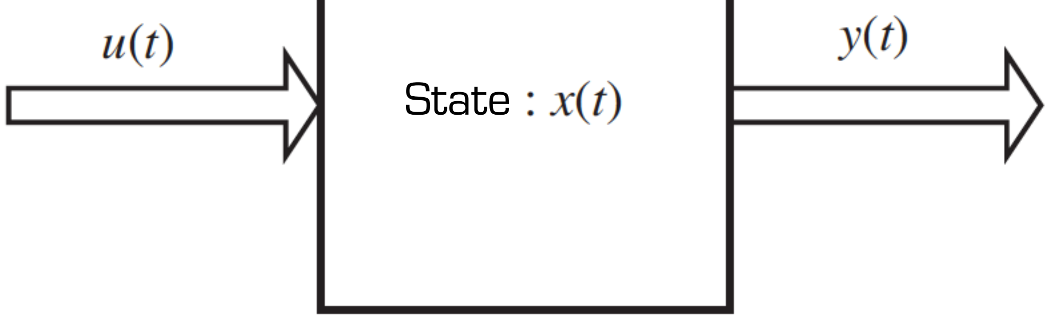
Recap

- State-space Representation
- Linearization
  - Approximation of nonlinear functions
  - Jacobian

Linear and Time-Invariant System



$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

$$x(0) = x_0$$

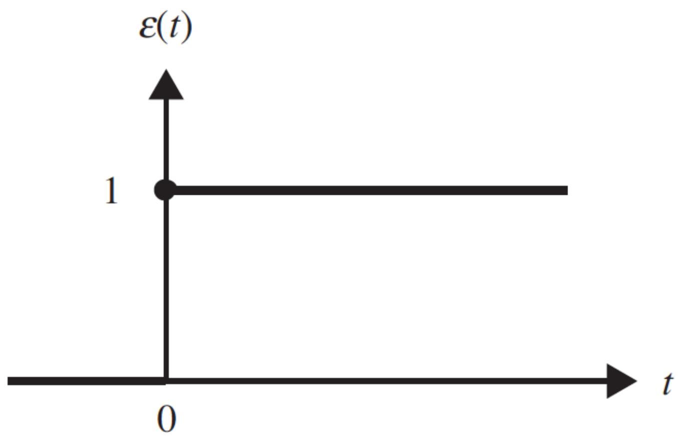
Lecture Overview

- Convolution integral and Impulse response
- Introduction to the Laplace Transform

Unit Step Function

- The unit step function is defined as

$$\varepsilon(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

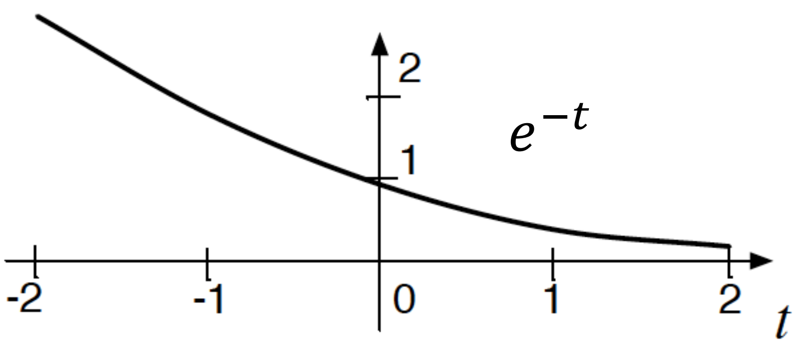


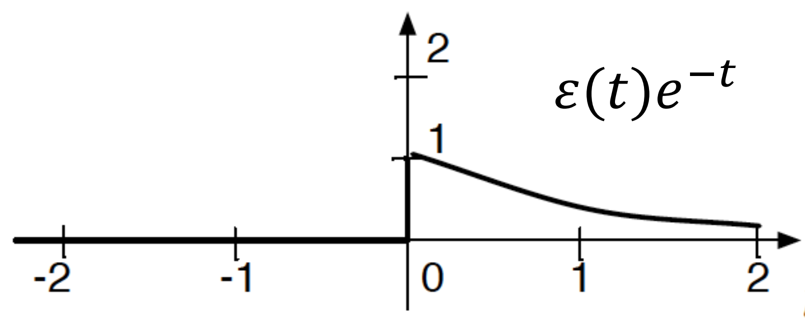
Utility of Unit Step Function

- Extracting part of another signal.
- For example, the piecewise-defined signal $x(t)$

$$x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

can be written as $x(t) = \varepsilon(t)e^{-t}$





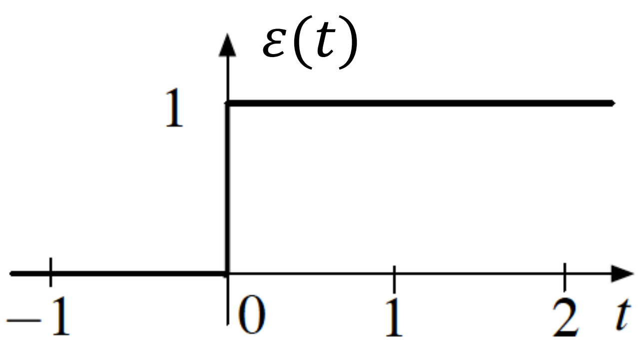
Utility of Unit Step Function

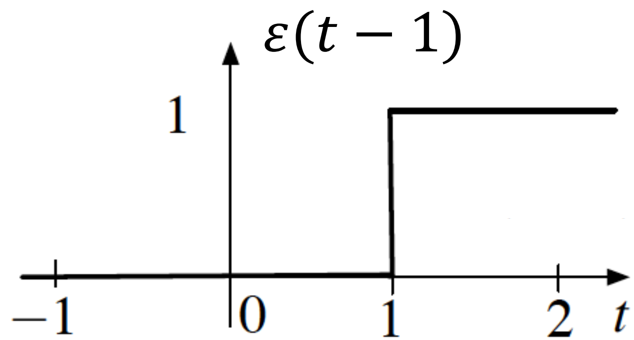
- Combinations of unit step functions to create other signals.

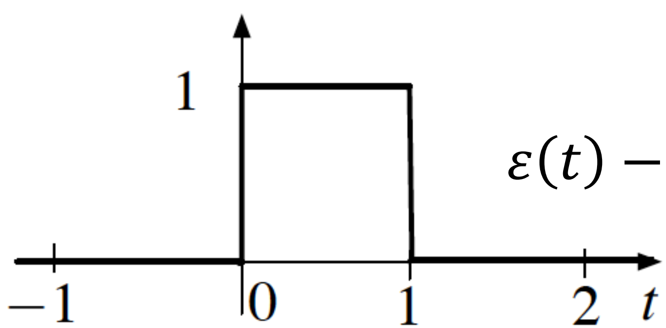
$$x(t) = \begin{cases} 0, & t \geq 1 \\ 1, & 0 \leq t < 1 \\ 0, & t < 0 \end{cases}$$

can be written as

$$x(t) = \varepsilon(t) - \varepsilon(t-1)$$





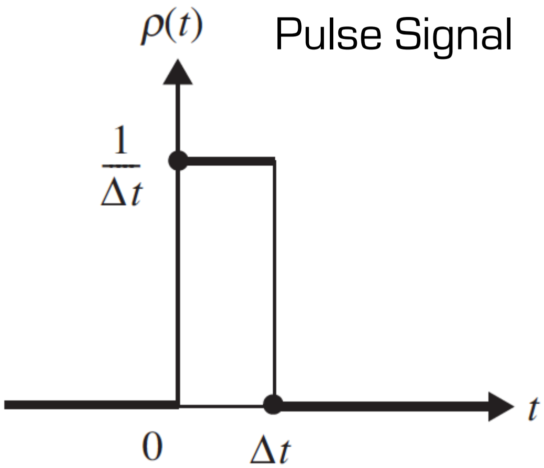


$\varepsilon(t) - \varepsilon(t-1)$

Impulse Signal

- Dirac delta function or unit impulse is an idealization of a pulse signal that
  - is very large near $t = 0$
  - is very small away from $t = 0$
  - has integral 1

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



- The exact shape does not matter
- Δt is small

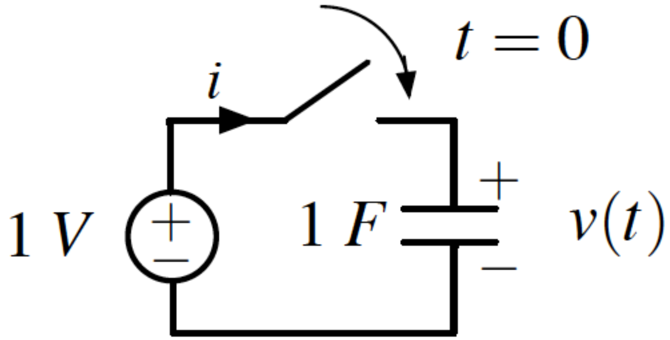
$$\delta(t) = \lim_{\Delta t \to 0} \rho(t)$$

$$\rho(t) = \begin{cases} 0 & t < 0 \\ 1/\Delta t & t \in [0, \Delta t) \\ 0 & t \geq \Delta t \end{cases}$$

Physical Interpretation

- Impulse functions are used to model physical signals
  - That act over short time intervals
  - Whose effect depends on the integral of the signal

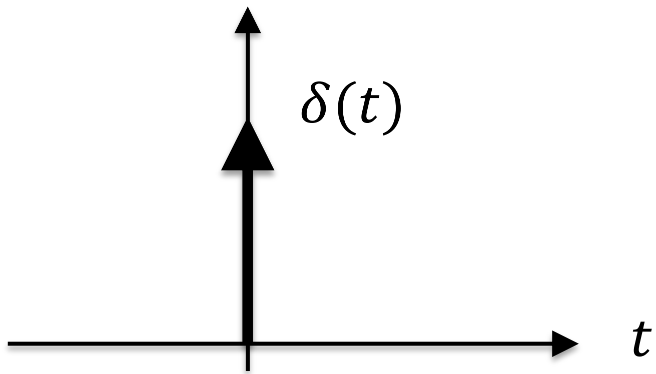
- **Examples**: hammer blow, bat hitting ball instantaneously, rapid charging of a capacitor



- $i(t)$ is very large, for a very short time
- A unit charge is transferred to the capacitor almost instantaneously
- $v(t)$ increases to 1V almost instantaneously

Impulse Signal

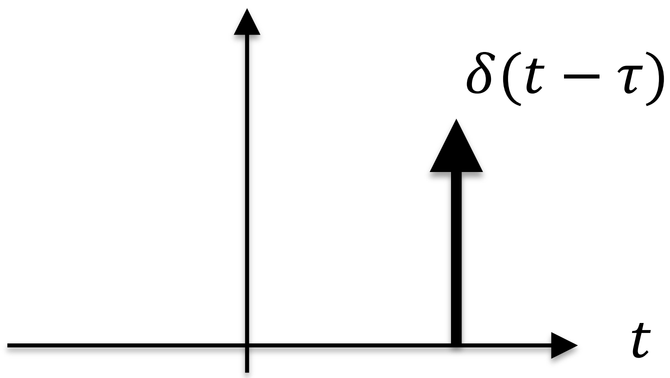
- Impulse signal $\delta(t)$ is plotted as



- Delta function is not a function
- Conceptually $\delta(t) = 0$ for $t \neq 0$, infinite at $t = 0$, but this does not make sense mathematically
- It only makes mathematical sense when shown inside an integral sign
- Pulse with an infinitesimal duration

Impulse Signal

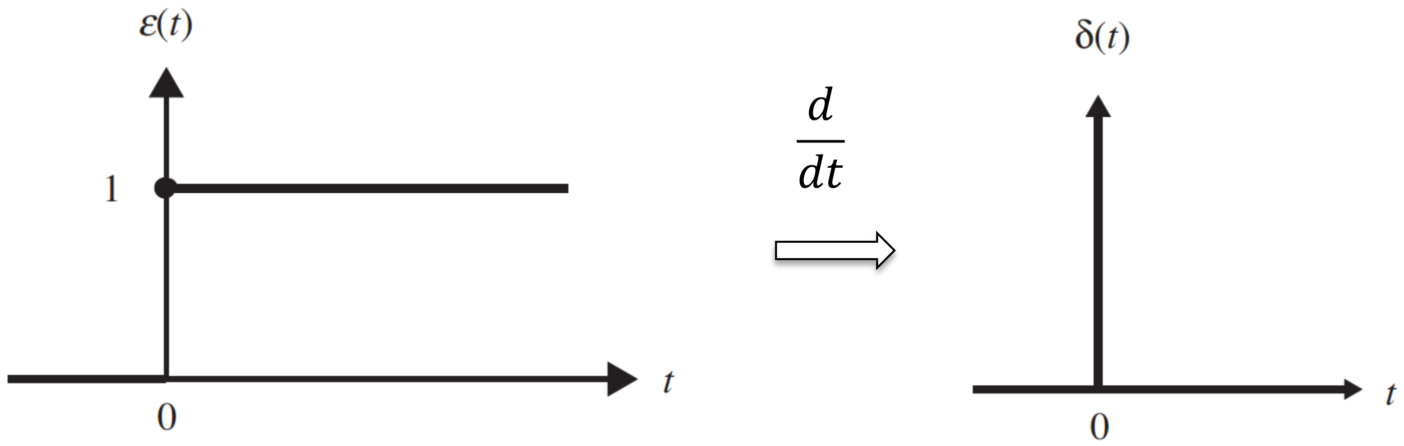
- Time shift
- The signal $\delta_\tau(t)$ = $\delta(t$- $\tau)$ is a function with impulse at time t = τ



Formal properties

- Derivative of a function with discontinuities generate impulse at each jump
- Derivative of unit step function is impulse function

$$\delta(t) = \frac{d\varepsilon(t)}{dt}$$



Formal properties

- Formally we define $\delta(t)$ by the property

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

provided that f is continuous at $t = 0$

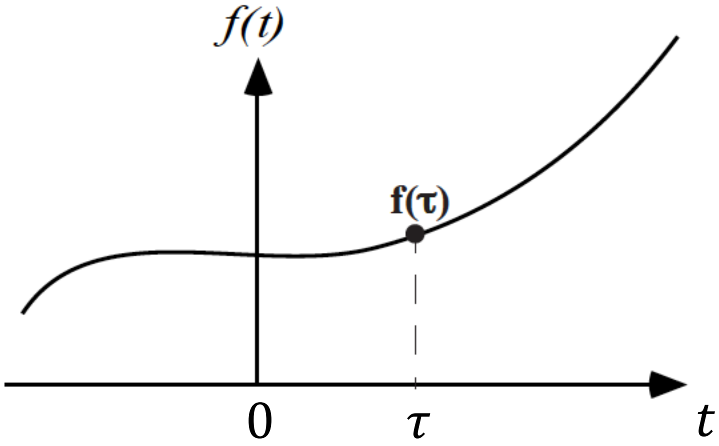
- **Underlying assumption**: $\delta(t)$ acts over a very small time interval, over which:

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

Formal properties

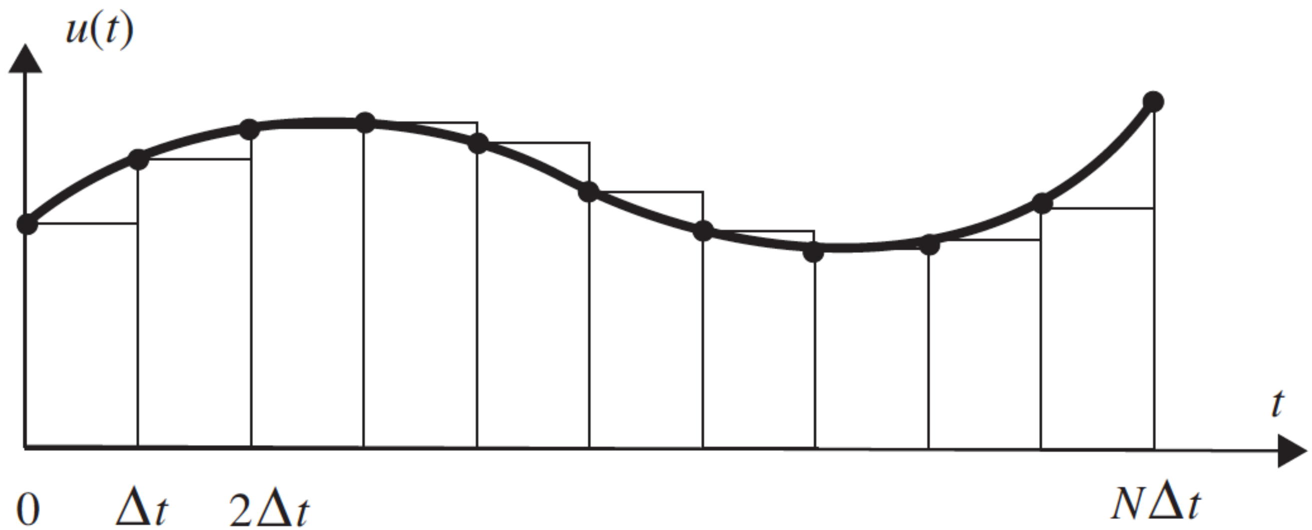
- For f continuous at $t = \tau$,

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t) \mathrm{d}t = f(\tau)$$



Input signal as a sample of itself

- Representation of a signal as a weighted integral of impulse functions

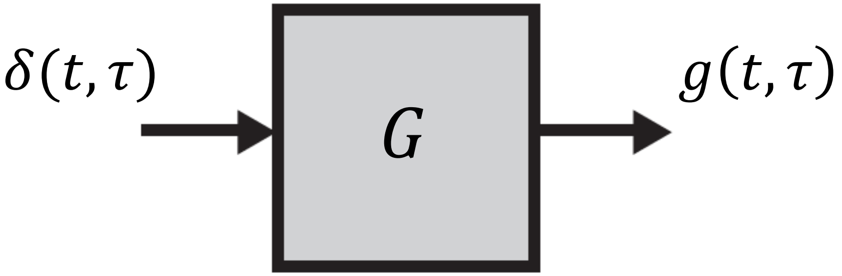


$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,\mathrm{d}\tau$$

Impulse Response

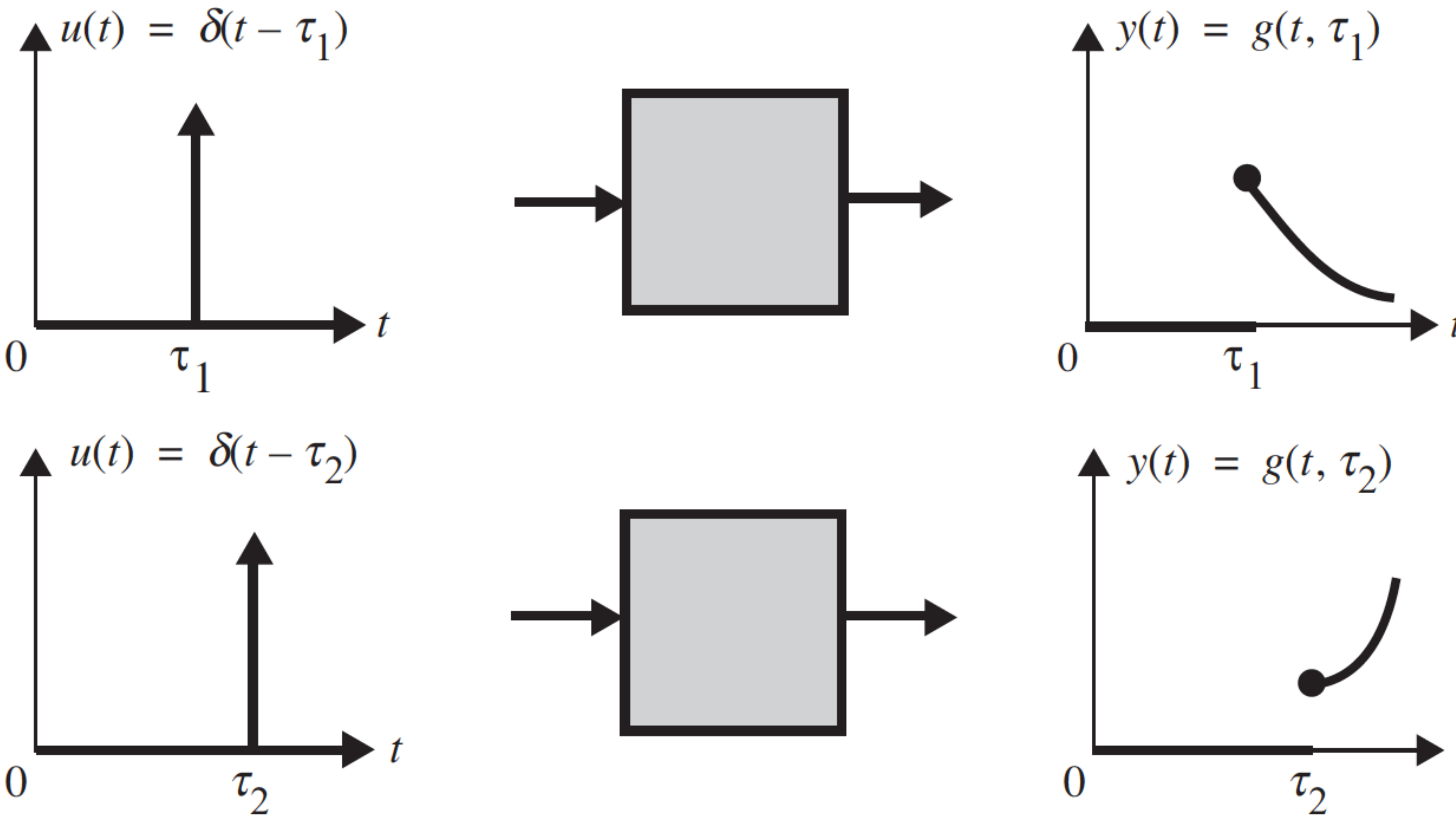
- The impulse response of a linear system $g(t, \tau)$ is the output of the system at time t to an impulse at time τ. This can be written as

$$g(t, \tau) = G(\delta(t, \tau))$$



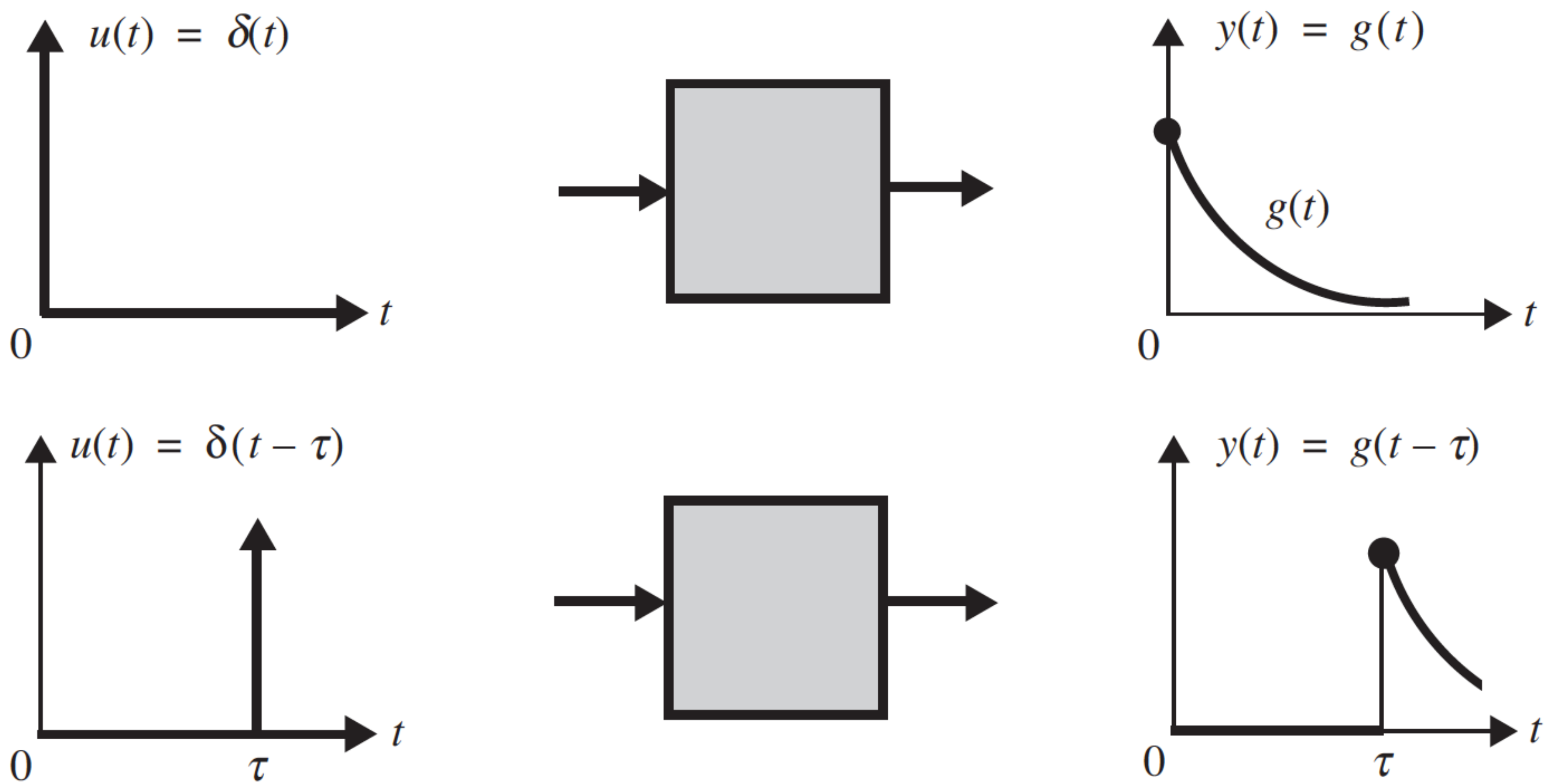
Impulse Response

- Linear, Time-variant System



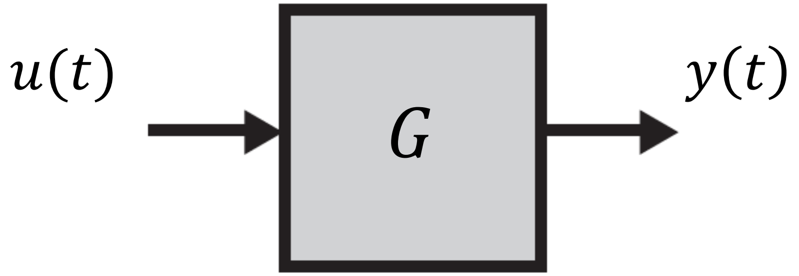
Impulse Response

- Linear, Time-invariant System
- Delaying the input and output by a time τ should produce the same response



Output of an LTI System

- We would like to determine an expression for the output y(t) of a linear and time-invariant system for a given input u(t)



$$y(t) = G\big(u(t)\big) = G\left(\int_{-\infty}^{\infty} u(\tau)\delta(t,\tau)d\tau\right) = \int_{-\infty}^{\infty} u(\tau)G(\delta(t,\tau))d\tau$$

Extended linearity

$$= \int_{-\infty}^{\infty} u(\tau)g(t,\tau)d\tau$$

Impulse Response

The Convolution Integral

- For a continuous-time linear dynamical system, the output is given by the superposition integral:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t,\tau)d\tau$$

- For a continuous-time LTI dynamical system, the output is represented by the convolution integral as follows:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)g(t-\tau)d\tau := u(t) * g(t)$$

- Time-invariance

$$g(t,\tau) = g(t-\tau)$$

- Causal: future values
- Initially at rest (no output for $t<0$)

$$y(t) = \int_{0}^{t} u(\tau)g(t-\tau)d\tau$$

Take home message

- The input signal can be decomposed into a set of impulses, each of which can be viewed as a scaled and shifted delta function.
- The output resulting from each impulse is a scaled and shifted version of the impulse response.
- The overall output signal can be found by adding these scaled and shifted impulse responses.

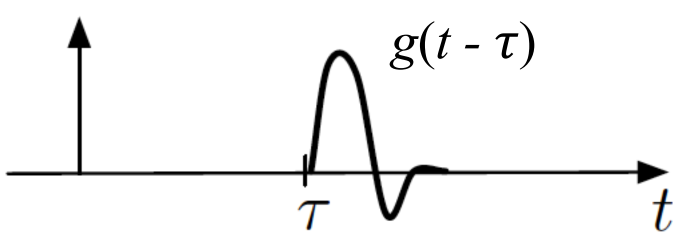
- In other words, if we know a system's impulse response, then we can calculate what the output will be for any possible input signal.

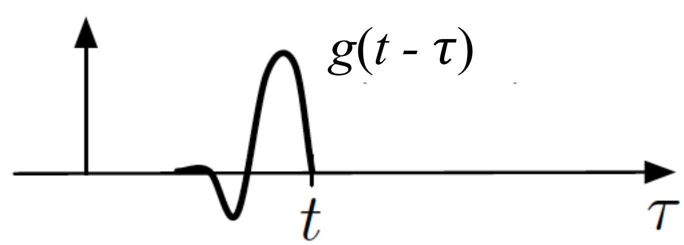
Graphical Interpretation

- The output is the integral of all of these responses:

$$y(t) = \int_{-\infty}^{\infty} u(\tau) g(t - \tau) d\tau$$

- Let's look at the integral:
  - $g(t, \tau) = g(t - \tau)$ is the impulse response delayed to time τ
  - If we consider $g(t - \tau)$ to be a function of τ, then $g(t - \tau)$ is delayed to time t and reversed



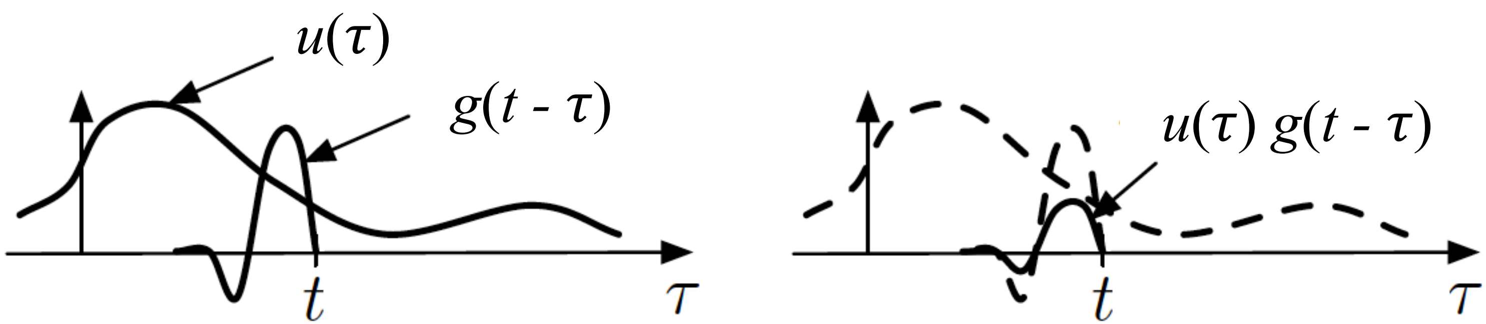


Graphical Interpretation

- The output is the integral of all of these responses:

$$y(t) = \int_{-\infty}^{\infty} u(\tau) g(t - \tau) d\tau$$

- Let's look at the integral:
  - This is multiplexed point by point with the input $u(t)$
  - Then integrate over to find $y(t)$ for the current time t



Graphical Interpretation

- Graphically, to find $y(t)$:
  - Flip impulse response backwards in time: yields $g(-\tau)$
  - Drag to the right over t : yields $g(-(\tau - \mathrm{t}))$
  - Multiply pointwise by input : yields $u(\tau)\ g(t - \tau)$
  - Integrate over to get $y(t)$

Graphical Interpretation

1. Express each function in terms of a dummy variable τ.
2. Reflect one of the functions: $g(\tau) \to g(-\tau)$.
3. Add a time-offset, t, which allows $g(t-\tau)$ to slide along the τ-axis.
4. Start t at $-\infty$ and slide it all the way to $+\infty$. Wherever the two functions intersect, find the integral of their product. In other words, compute a sliding, weighted-sum of function $f(\tau)$, where the weighting function is $g(-\tau)$.

The resulting waveform (not shown here) is the convolution of functions f and g.

If $f(t)$ is a unit impulse, the result of this process is simply $g(t)$. Formally:

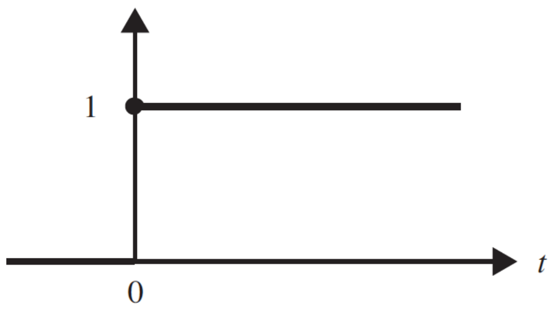
$$\int_{-\infty}^{\infty} \delta(\tau) g(t-\tau)\, d\tau = g(t)$$

https://en.wikipedia.org/wiki/Convolution

Example 1

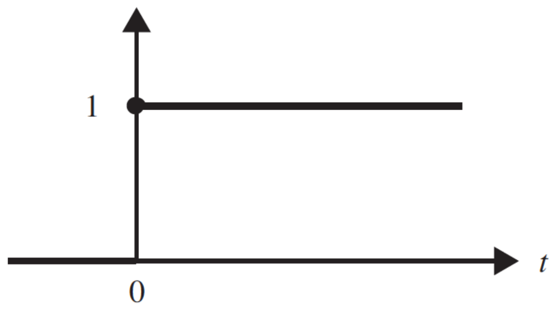
Impulse Response

$g(t) = \varepsilon(t)$



Input Signal

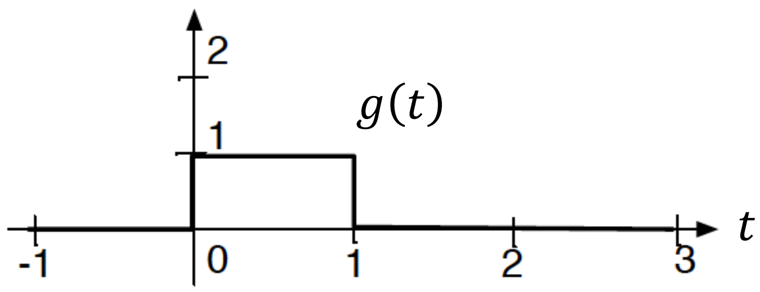
$u(t) = \varepsilon(t)$



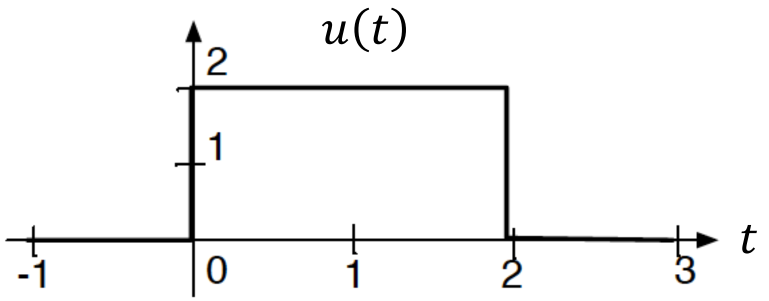
$$y(t) = u(t) * g(t) = \int_0^t u(\tau)g(t-\tau)d\tau = \int_0^t \varepsilon(\tau)\varepsilon(t-\tau)d\tau$$

Example 2

Impulse Response

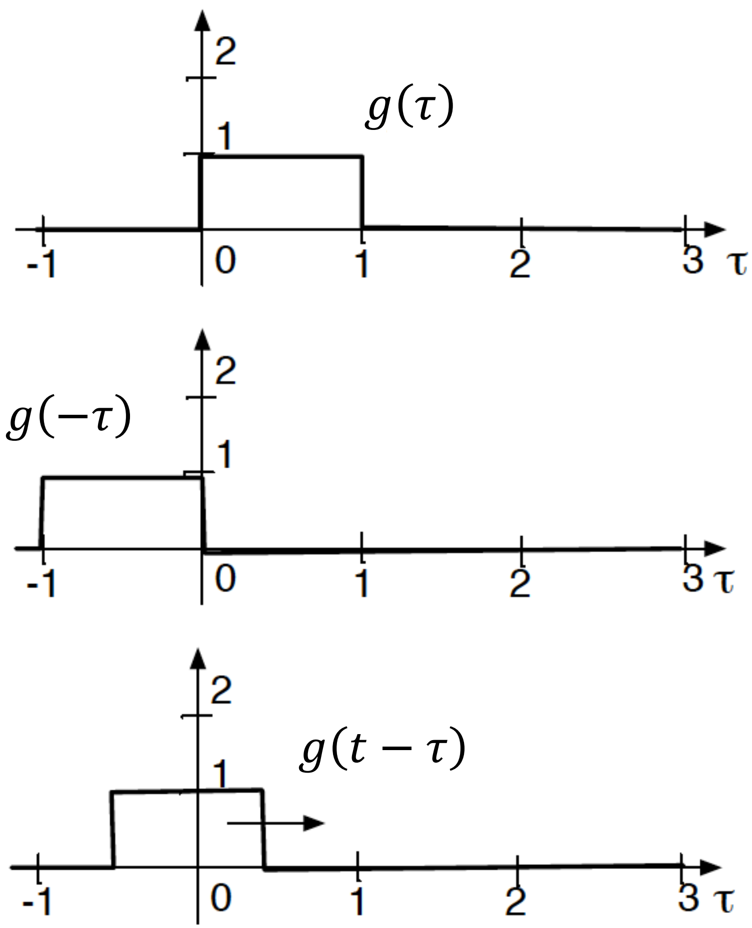


Input Signal

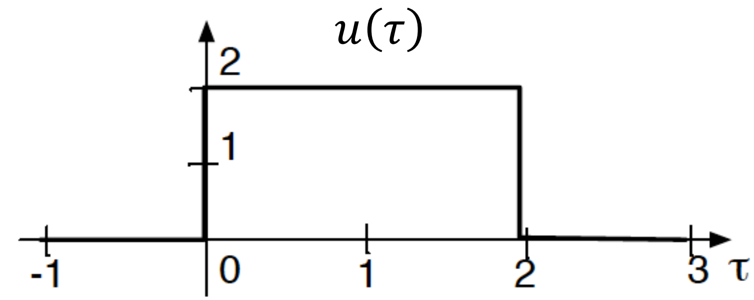


Example 2

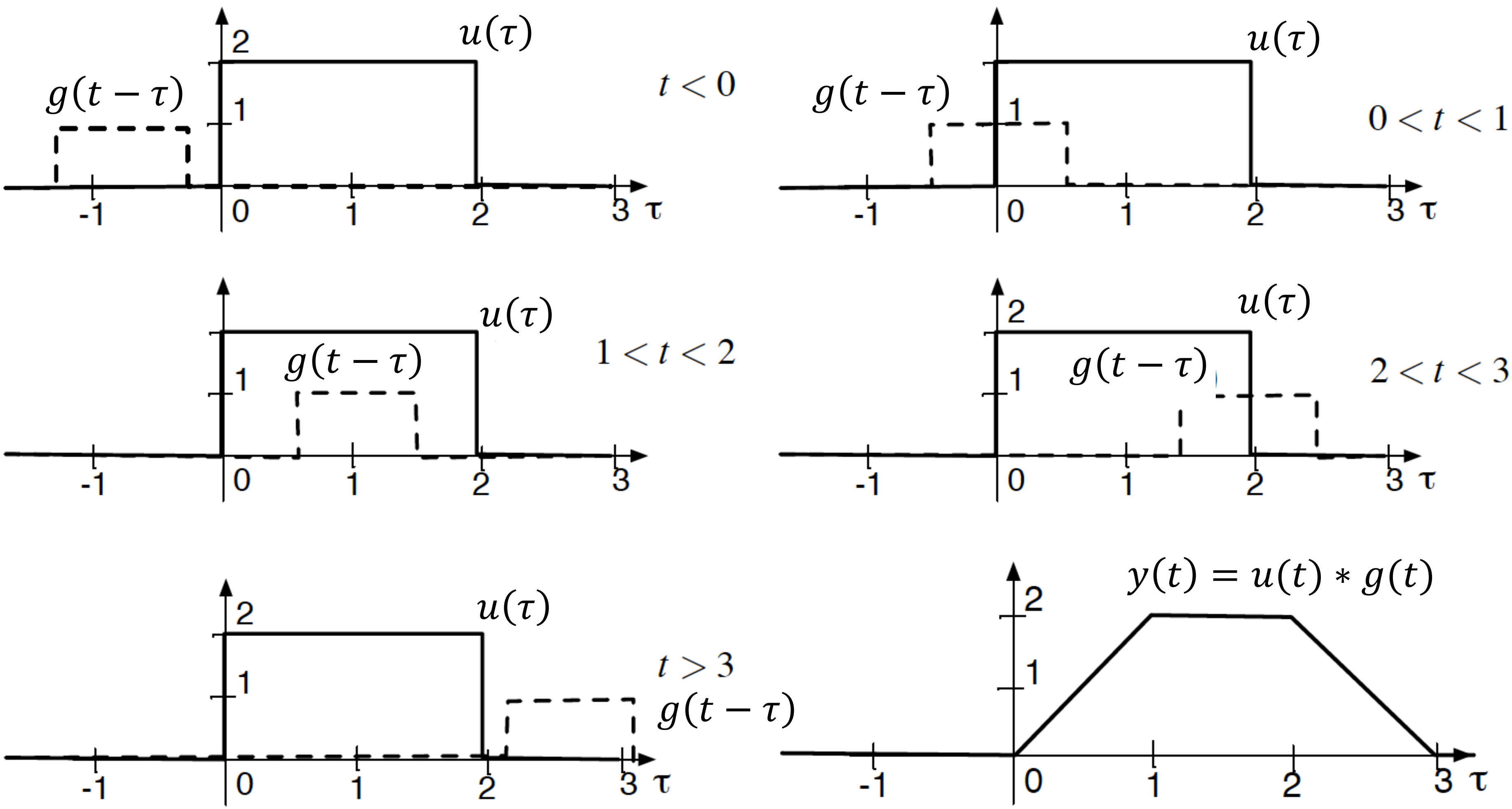
Impulse Response



Input Signal



Example 2



Properties of Convolution

- **Commutative Property**

$$g(t) * u(t) = u(t) * g(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

- If we make the substitution $\tau_1 = t - \tau$, then $\tau = t - \tau_1$, and $d\tau = -d\tau_1$

$$g(t) * u(t) = \int_t^0 g(t-\tau_1)u(\tau_1)(-d\tau_1) = \int_0^t u(\tau_1)g(t-\tau_1)(d\tau_1)$$

$$= u(t) * g(t)$$

- Practically, if we have two signals to convolve, we can choose either to be the signal we hold constant and the other to **flip and drag**.

Properties of Convolution

- **Associative Property**

$$u(t) * [g_1(t) * g_2(t)] = [u(t) * g_1(t)] * g_2(t)$$

- Combined with commutative property, associative property means that we can perform the convolutions in any order

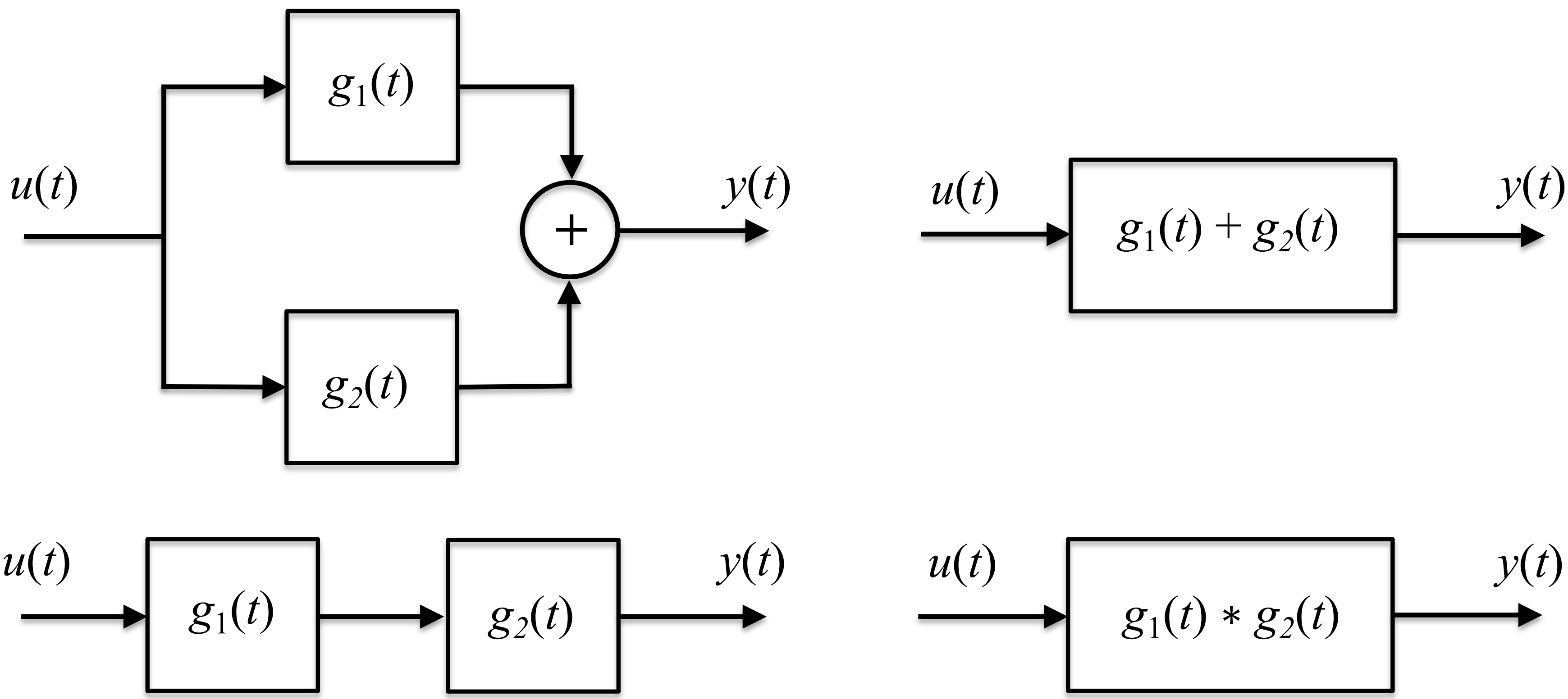
- **Distributive Property (Linearity)**

$$u(t) * [g_1(t) + g_2(t)] = u(t) * g_1(t) + u(t) * g_2(t)$$

- All three properties together mean that there is an algebra of signals
  - Addition is like arithmetic or ordinary algebra
  - Multiplication is replaced by convolution

Properties of Convolution Systems

- Composition of convolution systems corresponds to convolution of impulse responses
- Many operations can be written as convolutions and they all commute: differentiation, integration, delay etc.



Properties of Convolution Systems

- Convolution systems are time-invariant
- If we shift the input signal $u(t)$ by τ, i.e. apply the input

$$u_1(t) = u(t - \tau)$$

to the system, the output is

$$y_1(t) = y(t - \tau)$$

- In other words, convolution systems commute with delay

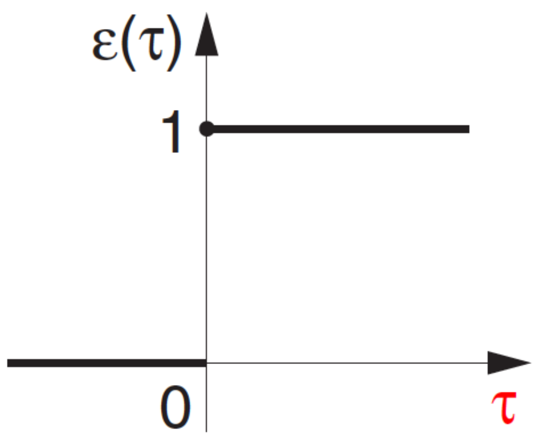
Example 3

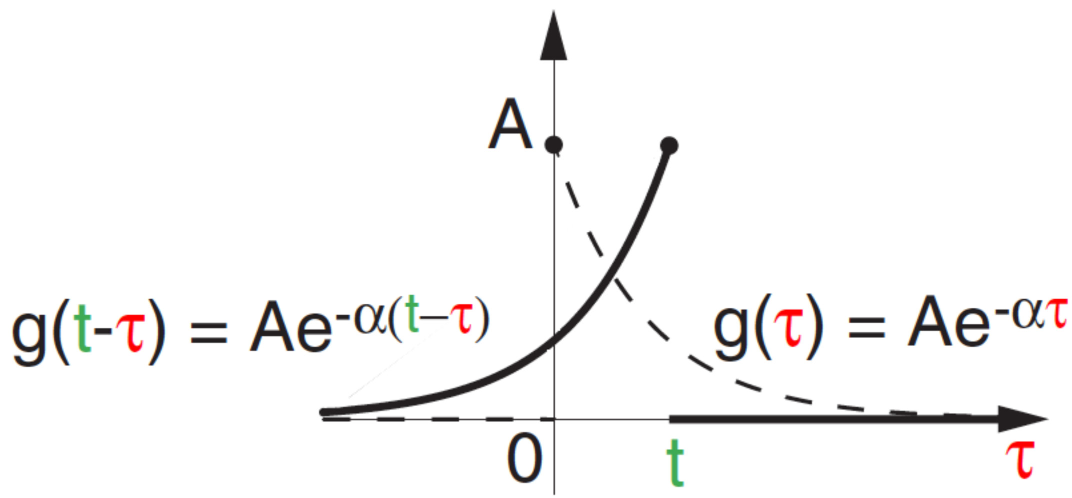
- The impulse response of an LTI dynamical system is given by:

$$g(t) = Ae^{-\alpha t}$$

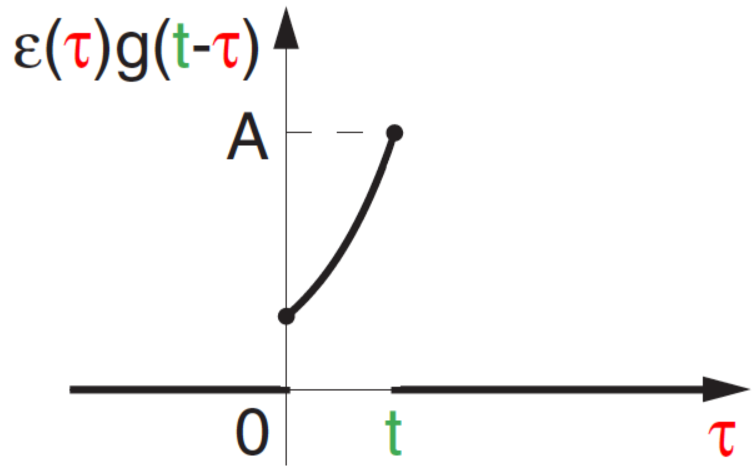
Find the unit step response (output for the input signal represented by a unit step function).

$$y(t) = \varepsilon(t) * g(t) = \int_0^t \varepsilon(\tau) g(t-\tau) \mathrm{d}\tau$$





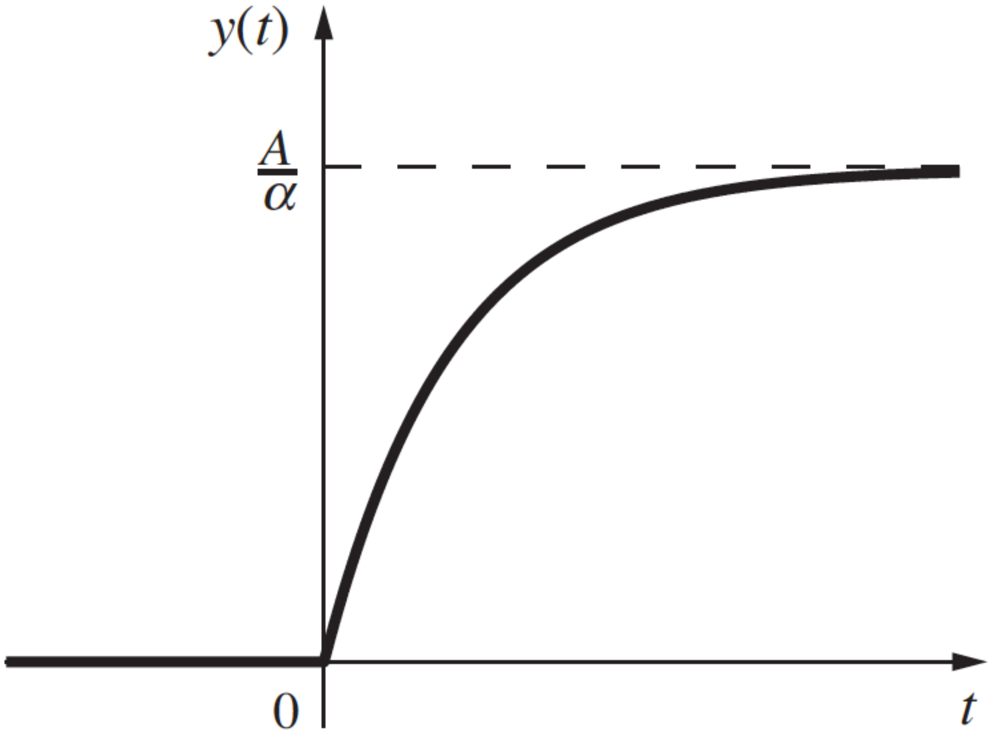
Example 3



- $t < 0$ $\quad \int_0^t 0 \, d\tau = 0$
- $t \geq 0$ $\quad \int_0^t A e^{-\alpha(t-\tau)} d\tau = A e^{-\alpha t} \int_0^t e^{\alpha\tau} d\tau = A e^{-\alpha t} \frac{1}{\alpha} e^{\alpha\tau} \Big|_0^t$

$$= \frac{A}{\alpha} e^{-\alpha t} (e^{\alpha t} - 1) = \frac{A}{\alpha} (1 - e^{-\alpha t})$$

Example 3



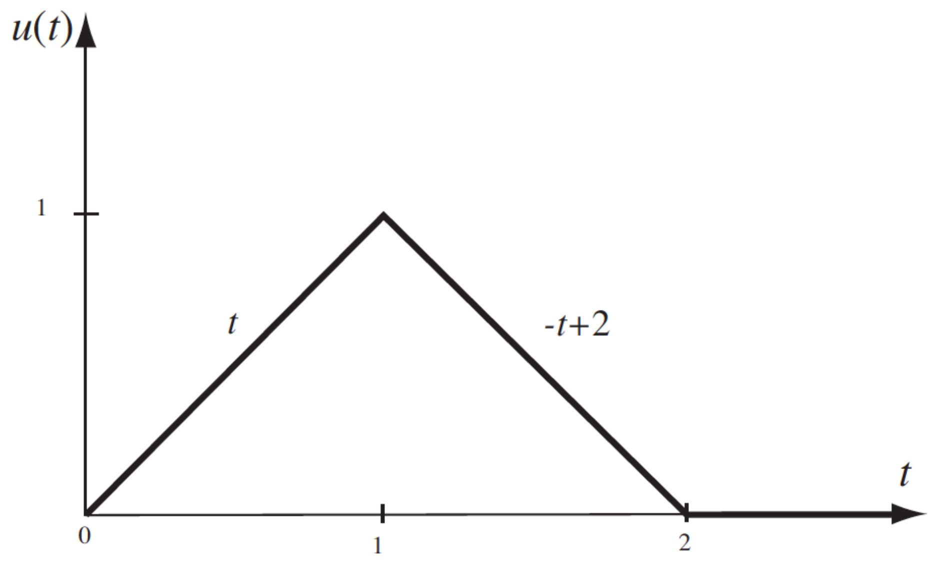
$$y(t) = \frac{A}{\alpha}(1 - e^{-\alpha t})$$

Example 4

- The impulse response of an LTI dynamical system is given by:

$$g(t) = \varepsilon(t)$$

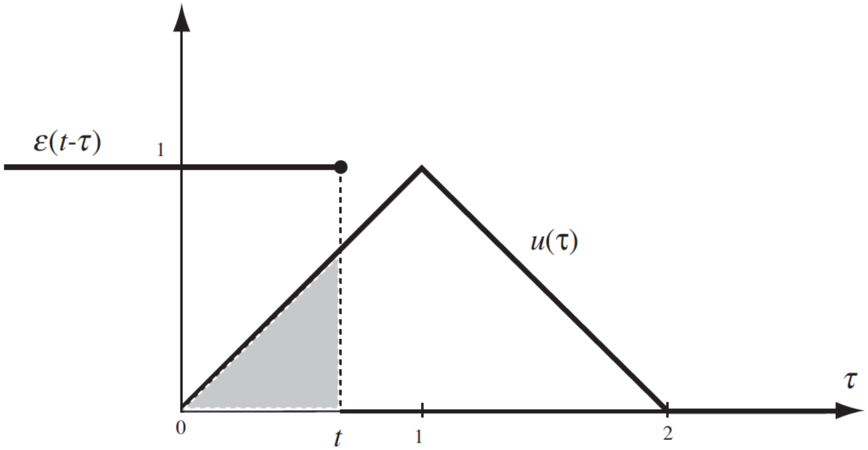
Find the output signal in response to the following input signal.



Example 4

For $0 \le t < 1$

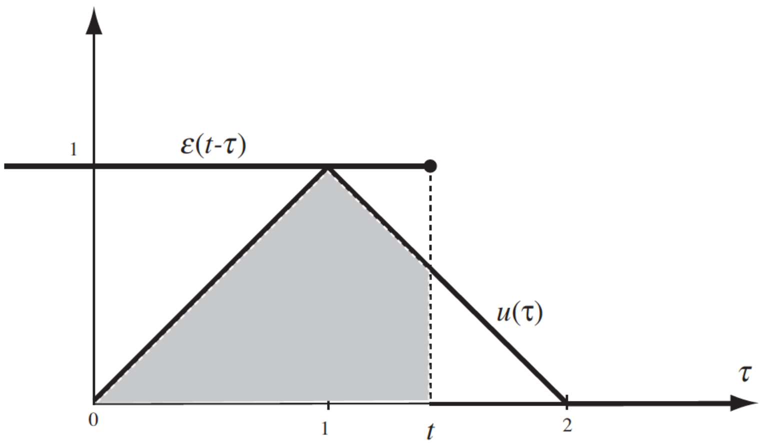
$$\int_0^t u(\tau)\varepsilon(t-\tau)\,d\tau = \int_0^t \tau \cdot 1 \cdot d\tau = \frac{1}{2}t^2$$



For $1 \le t < 2$

$$\int_0^t u(\tau)\varepsilon(t-\tau)d\tau = \int_0^1 \tau \cdot 1 \cdot d\tau + \int_1^t (-\tau + 2) \cdot 1 \cdot d\tau$$

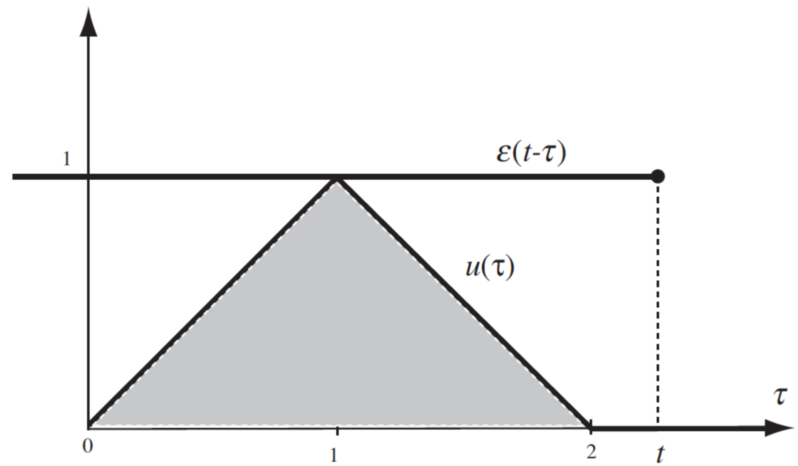
$$= \frac{1}{2} + \left(-\frac{1}{2}\tau^2 + 2\tau\right)\Big|_1^t = -\frac{1}{2}(t^2 - 4t + 2)$$



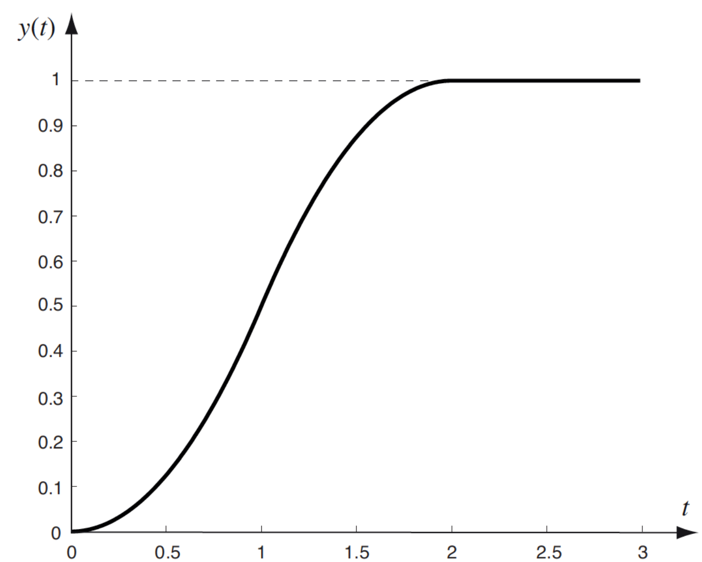
Example 4

For $t \geq 2$

$$\int_0^t u(\tau)\varepsilon(t-\tau)\,\mathrm{d}\tau = 2 \cdot \frac{1}{2} = 1$$



$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{2}t^2 & 0 \leq t < 1 \\ -\frac{1}{2}t^2 + 2t - 1 & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases}$$



Example 5

- Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & elsewhere \end{cases}$$

$$g(t) = \delta(t+2) + 2\delta(t+1)$$

- Answer:

Example 5

- Determine and sketch the convolution of the following two signals:

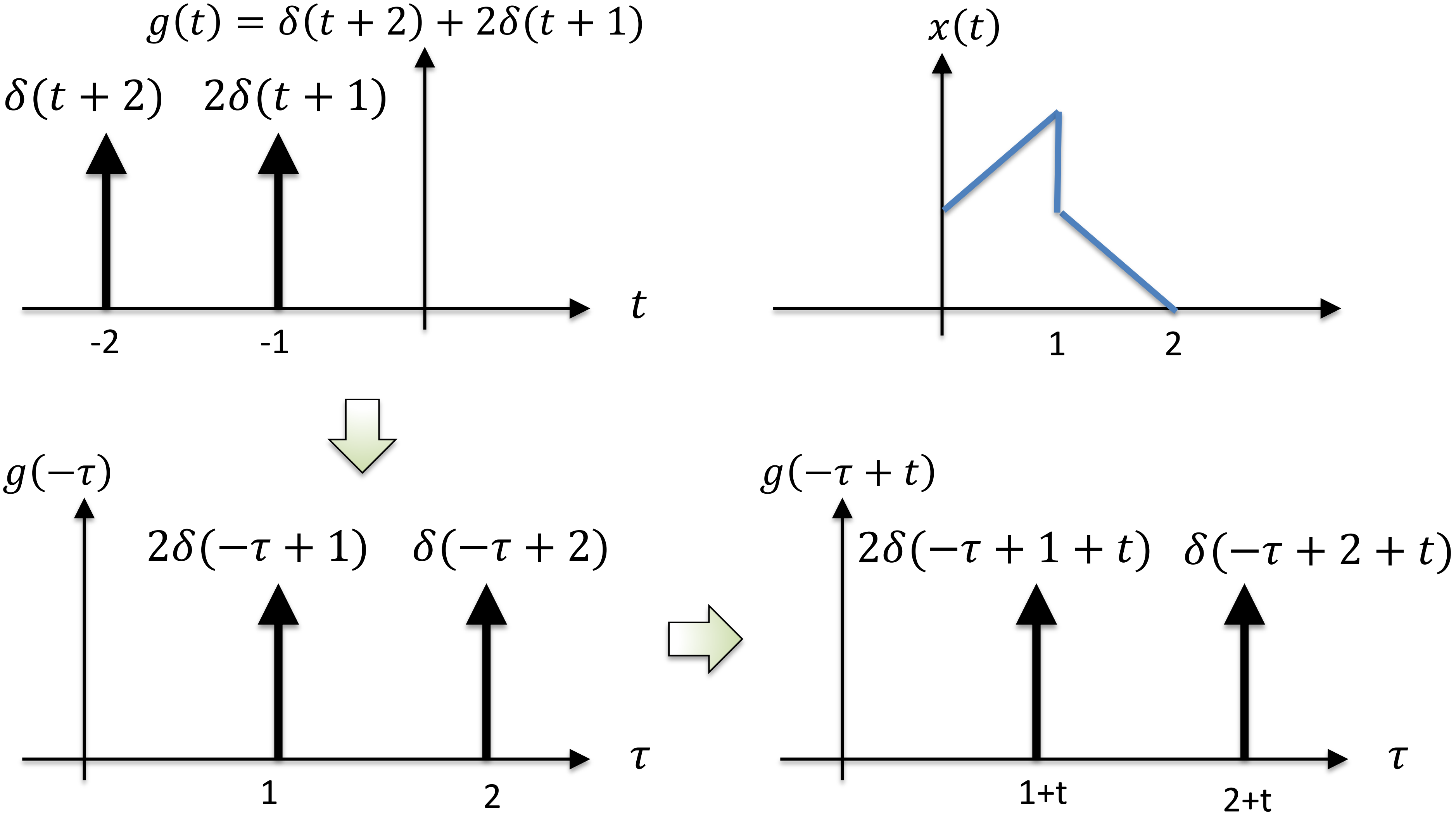
$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & elsewhere \end{cases}$$

$$g(t) = \delta(t+2) + 2\delta(t+1)$$

- Answer:

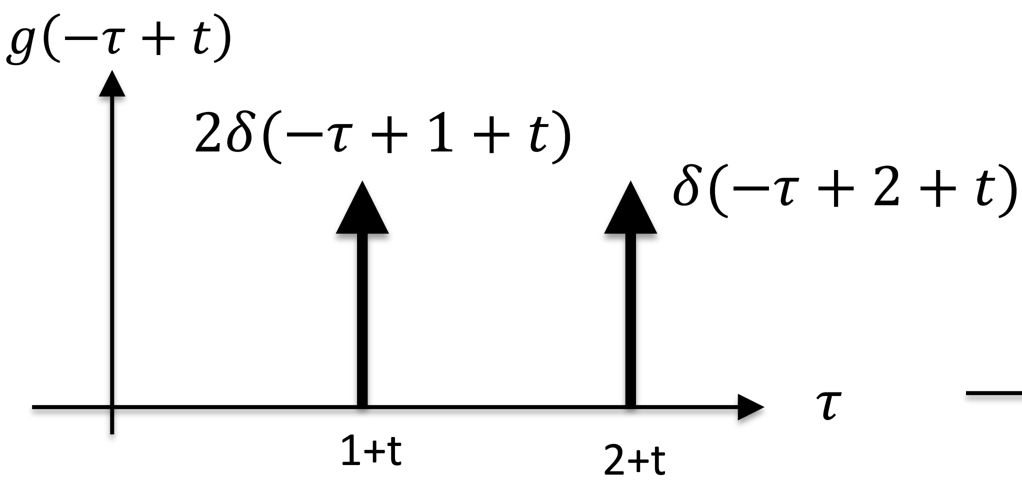
$$y(t) = \begin{cases} t+3, & -2 < t \le -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & elsewhere \end{cases}$$

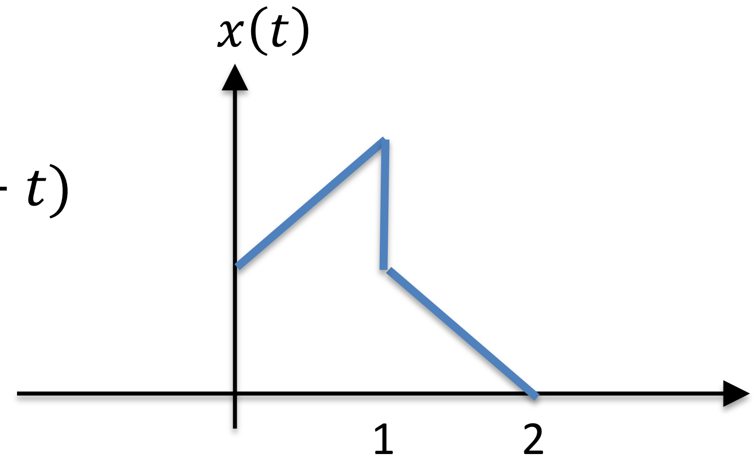
Example 5



$g(t) = \delta(t+2) + 2\delta(t+1)$
$\delta(t+2)$
$2\delta(t+1)$
t
-2
-1
$x(t)$
1
2
$g(-\tau)$
$2\delta(-\tau+1)$
$\delta(-\tau+2)$
τ
1
2
$g(-\tau+t)$
$2\delta(-\tau+1+t)$
$\delta(-\tau+2+t)$
τ
1+t
2+t

Example 5





$$\int_{-\infty}^{\infty} \delta(t - \tau) f(t) \mathrm{d}t = f(\tau)$$

Finding impulse response from unit step response

- It is experimentally challenging to generate impulse function.

- Impulse response can be found from the unit step response.

  - **Unit step response**

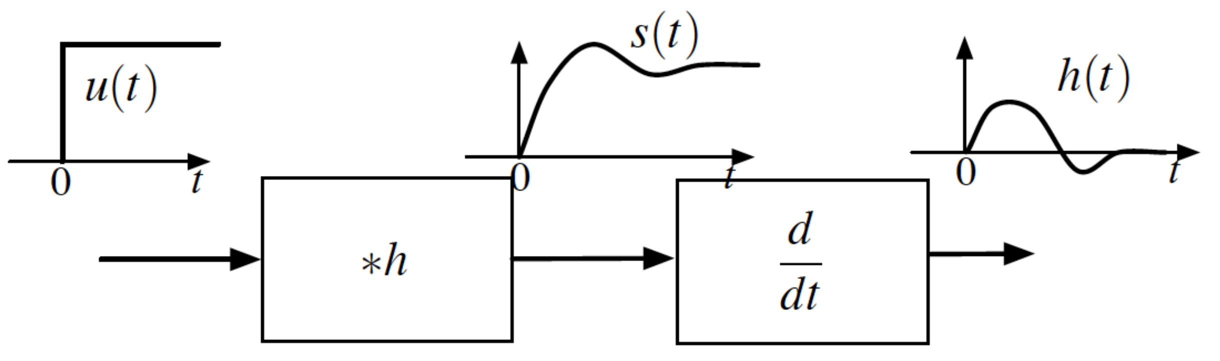
$$s(t) = \int_{-\infty}^{\infty} \varepsilon(\tau) g(t-\tau) d\tau = \int_{0}^{t} g(t-\tau) d\tau = \int_{0}^{t} g(\tau) d\tau$$

  - **Impulse response**

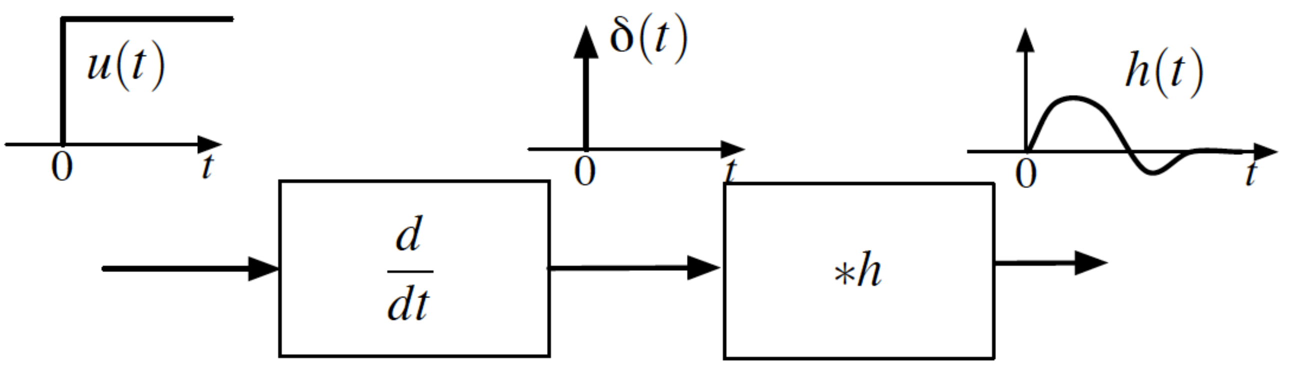
$$\frac{ds(t)}{dt} = g(t)$$

Finding impulse response from unit step response

- The impulse response is determined by differentiating the step response



- To show this, commute the convolution system and the differentiator



Perspective: Finding the output for a given input

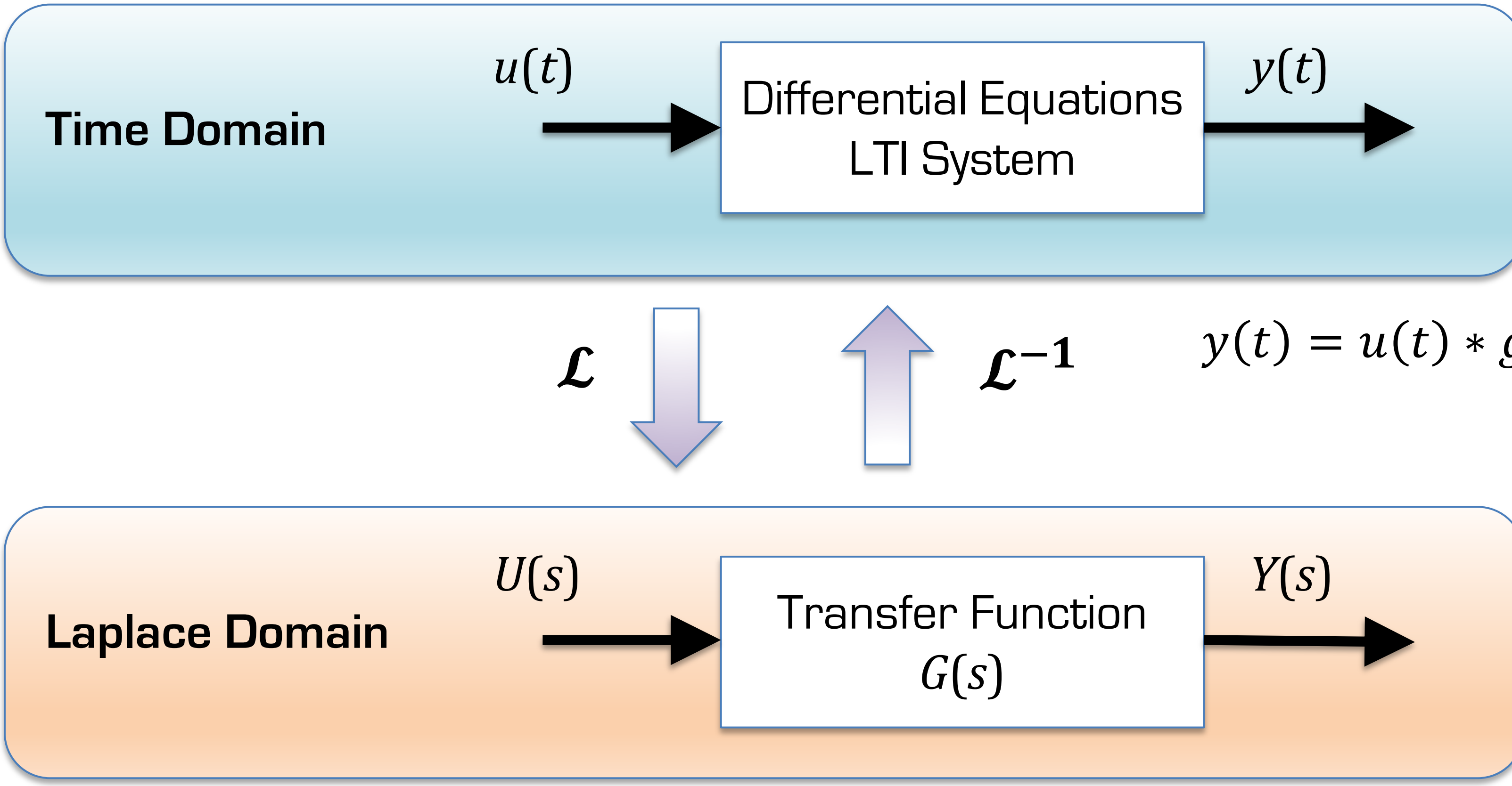
- **Option 1**: Solving the state equation

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- **Option 2**: Impulse response and convolution integral

- **Option 3**: Laplace transform and transfer function

Projection between Time and Laplace Domains



$$y(t) = u(t) * g(t)$$

$$Y(s) = U(s)G(s)$$

Complex Numbers

- Notation

$$s = a + jb \qquad j = \sqrt{-1}$$

- Complex Conjugate (reflection about the real axis)

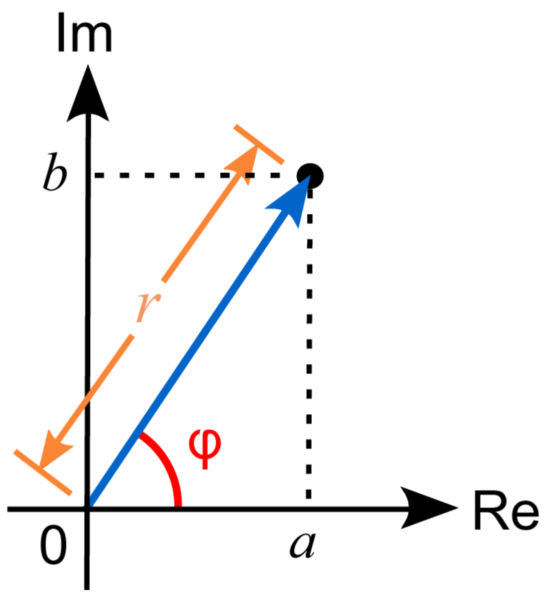
$$\bar{s} = a - jb$$

- Complex Algebra

$$\frac{1}{s} = \frac{a}{a^2 + b^2} - j\frac{b}{a^2 + b^2}$$

- Absolute value

$$r = |s| = \sqrt{a^2 + b^2}$$



The exponential function

- Taylor Series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$$

- Euler's identities

$$e^{j\omega} = \cos\omega + j\sin\omega$$

$$e^{-j\omega} = \cos\omega - j\sin\omega$$

- Limits

$$\lim_{x\to\infty} xe^{-x} = 0 \quad if\ x\ is\ real$$

$$\lim_{t\to\infty} e^{-st} = 0 \quad if\ the\ real\ part\ of\ s\ is\ positive$$

Laplace Transform

- For a given function $f(t)$ with $f(t) = 0$ for $t < 0$, Laplace transform of this function is defined as follows:

$$F(s) = \mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)e^{-st}dt$$

s is a complex variable

$$s = a + jb$$

$$f(t) \xrightarrow{\quad\mathcal{L}\quad} F(s)$$

- **Inverse Laplace Transform**

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

Existence of Laplace Transform

- The Laplace transform of a function $f(t)$ exists if the Laplace integral converges

- The integral will converge if $f(t)$ is piecewise continuous in every finite interval in the range $t > 0$ and if $f(t)$ is of exponential order as t approaches infinity.

- A function is said to be of exponential order if a real, positive constant σ exists such that the function

$$e^{-\sigma t}|f(t)|$$

approaches zero as t approaches infinity. If the limit approaches zero for σ greater than σ_c and approaches infinity for σ less than σ_c the value σ_c is called the **abscissa of convergence**.

Existence of Laplace Transform

- Abscissa of convergence is equal to zero for functions such as

$$t, \sin \omega t, t \sin \omega t$$

- Abscissa of convergence is equal to -c for functions such as

$$e^{-ct}, te^{-ct}, e^{-ct} \sin \omega t$$

- In the case of functions that increase faster than the exponential function, it is impossible to find suitable values of the abscissa of convergence

$$e^{t^2}, te^{t^2}$$

- Nevertheless, the following function does possess a Laplace transform

$$f(t) = \begin{cases} e^{t^2}, & 0 \le t \le T < \infty \\ 0, & t < 0, T < t \end{cases}$$

- **All signals that can be physically generated have Laplace transforms.**